A Study on Cooling and Trapping of a 2 Level Atom Under Artificial Gauge Field using 3 Highly Coupled Quantum Harmonic Oscillators

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The behaviour of a single 2 level atom in a rotating cavity or equivalently under artificial Gauge Field has studied using 3 highly coupled quantum harmonic oscillators. One of the oscillators corresponds to cavity field photons and the remaining two corresponds to x and y modes of the atom. It has observed that due to parity charge conservation in the system Linbladian can be block diagonalized into two blocks corresponding to two Hilbert Subspaces charachterized by the conserved parity charge [1]. Furthermore, due to block diagonalization of the Linbladian the system has 2 unique steady state living in corresponding Hilbert Subspace for each arbitrary set of parameters. We investigated the effect of rotation Ω_{rot}, cavity detuning Δ_c and initial density matrix ρ_0 on properties of mentioned steady states. Furthermore results of the semiclassical equations with stochastic noise to include first order quantum correlations has compared with the previous results.

I. MODEL

One can start with the general Jaynes–Cummings Hamiltonian in two dimensions for a two-level atom with a transverse pump in the z direction and a symmetric harmonic trapping potential after the rotating wave approximation and transformation to the pump frame [2]:

$$\hat{H}_p = \hat{H}_{\mathrm{cm}} + \hat{H}_{\mathrm{cav}} + \hat{H}_{\mathrm{at}} + \hat{H}_{\mathrm{int}} + \hat{H}_{\mathrm{pump}} \tag{1}$$

Here,

$$\hat{H}_{\mathrm{cm}} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m\omega_{\mathrm{trap}}^2 \hat{r}^2 \tag{2}$$

$$\hat{H}_{\mathrm{cav}} = -\hbar\Delta_c \hat{a}^\dagger \hat{a} \tag{3}$$

$$\hat{H}_{\mathrm{at}} = -\hbar\Delta_a \hat{\sigma}_+ \hat{\sigma}_- \tag{4}$$

$$\hat{H}_{\mathrm{int}} = -i\hbar g f_c \left(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger \hat{\sigma}_-\right) \tag{5}$$

$$\hat{H}_{\mathrm{pump}} = \hbar\Omega_p \left(\hat{\sigma}_+ e^{ik_p z} + \hat{\sigma}_- e^{-ik_p z}\right) \tag{6}$$

Here $f_c = f_c(\hat{x}, \hat{y})$. The detunings are

$$\Delta_c = \omega_p - \omega_c \qquad \Delta_a = \omega_p - \omega_a \tag{7}$$

The corresponding Liouvillian is

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}_p, \hat{\rho}] + \mathcal{D}_c\hat{\rho} + \mathcal{D}_a\hat{\rho} \tag{8}$$

with dissipators

$$\mathcal{D}_c\hat{\rho} = \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{9}$$

$$\mathcal{D}_a\hat{\rho} = \Gamma\left(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right) \tag{10}$$

Setting $k_p z = 3\pi/2$, which is an arbitrary phase choice, the pump term becomes

$$\hat{H}_{\mathrm{pump}} = -i\hbar\Omega_p\left(\hat{\sigma}_+ - \hat{\sigma}_-\right) \tag{11}$$

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In the far-detuned or dispersive region,

$$|\Delta_a| \gg g, \kappa, \Omega_p \Gamma \tag{12}$$

the upper atomic state can be adiabatically eliminated. Thus,

$$\dot{\hat{\sigma}}_- \simeq 0 \qquad \hat{\sigma}_z \simeq -1 \tag{13}$$

Substituting the terms from Eq. (13) to the differential equation one can get considering Eq. (122) for $\hat{\sigma}_-$ operator the steady-state atomic operators can be found as [3],

$$\hat{\sigma}_- \simeq -\frac{g f_c \hat{a} + \Omega_p}{\Gamma - i\Delta_a} \tag{14}$$

and $(\hat{\sigma}_-)^\dagger = \hat{\sigma}_+$. The dispersive light shift, scattering rate, and effective pump strength are

$$U_0 = \frac{\Delta_a g^2}{\Delta_a^2 + \Gamma^2} \qquad \Gamma_0 = \frac{\Gamma g^2}{\Delta_a^2 + \Gamma^2} \qquad \eta_{\mathrm{eff}} = \frac{\Delta_a g \Omega_p}{\Delta_a^2 + \Gamma^2} \tag{15}$$

After adiabatic elimination, the effective Hamiltonian is

$$\hat{H}_{\mathrm{eff}} = \hat{H}_{\mathrm{cm}} - \hbar\Delta_{\mathrm{eff}}\hat{a}^\dagger\hat{a} + \hbar\eta_{\mathrm{eff}} f_c\left(\hat{a} + \hat{a}^\dagger\right) \tag{16}$$

Where the dispersive atom-cavity interaction produces a position dependent AC Stark shift. After adiabatic elimination of the atomic excited state, this shift appears as a position dependent effective cavity detuning,

$$\Delta_{\mathrm{eff}}(\hat{x}, \hat{y}) = \Delta_c - U_0 f_c^2(\hat{x}, \hat{y}) \tag{17}$$

The Lindblad dissipators are

$$\mathcal{D}_1\hat{\rho} = \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{18a}$$

$$\mathcal{D}_2\hat{\rho} = \Gamma_0\Big(2\hat{a} f_c(\hat{x},\hat{y})\hat{\rho} f_c^\dagger(\hat{x},\hat{y})\hat{a}^\dagger - \hat{a}^\dagger\hat{a}|f_c(\hat{x},\hat{y})|^2\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}|f_c(\hat{x},\hat{y})|^2\Big) \tag{18b}$$

The time evolution in the non-rotating pump frame is

$$i\hbar\partial_t\,|\psi(t)\rangle = \hat{H}_{\rm eff}\,|\psi(t)\rangle \tag{19}$$

We transform to a frame rotating around the z axis with angular frequency $\Omega_{\rm rot}$ [4]. The angular momentum operator is

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{20}$$

The corresponding rotation operator is

$$\hat{R}_z(t) = \exp\left[-\frac{i}{\hbar}\Omega_{\rm rot}\hat{L}_z t\right] \tag{21}$$

The non-rotating and rotating states are related by

$$|\psi(t)\rangle = \hat{R}_z(t)\,|\psi_\Omega(t)\rangle \tag{22}$$

Substitution into Eq. (19) gives

$$i\hbar\partial_t\left[\hat{R}_z(t)\,|\psi_\Omega(t)\rangle\right] = \hat{H}_{\rm eff}\hat{R}_z(t)\,|\psi_\Omega(t)\rangle \tag{23}$$

Expanding the time derivative gives

$$i\hbar\left(\dot{\hat{R}}_z\,|\psi_\Omega\rangle + \hat{R}_z\partial_t\,|\psi_\Omega\rangle\right) = \hat{H}_{\rm eff}\hat{R}_z\,|\psi_\Omega\rangle \tag{24}$$

Multiplying by $\hat{R}_z^\dagger$ from the left,

$$i\hbar\partial_t\,|\psi_\Omega\rangle = \left(\hat{R}_z^\dagger\hat{H}_{\rm eff}\hat{R}_z - i\hbar\hat{R}_z^\dagger\dot{\hat{R}}_z\right)|\psi_\Omega\rangle \tag{25}$$

Therefore, the rotating-frame Hamiltonian is

$$\hat{H}_\Omega = \hat{R}_z^\dagger\hat{H}_{\rm eff}\hat{R}_z - i\hbar\hat{R}_z^\dagger\dot{\hat{R}}_z \tag{26}$$

Since

$$\dot{\hat{R}}_z = -\frac{i}{\hbar}\Omega_{\rm rot}\hat{L}_z\hat{R}_z \tag{27}$$

we obtain

$$-i\hbar\hat{R}_z^\dagger\dot{\hat{R}}_z = -\Omega_{\rm rot}\hat{L}_z \tag{28}$$

Thus,

$$\hat{H}_\Omega = \hat{R}_z^\dagger\hat{H}_{\rm eff}\hat{R}_z - \Omega_{\rm rot}\hat{L}_z \tag{29}$$

The coordinate and momentum transformations are written compactly as

$$\hat{\mathbf{r}}_{\rm lab} = R(\Omega_{\rm rot}t)\hat{\mathbf{r}} \qquad \hat{\mathbf{p}}_{\rm lab} = R(\Omega_{\rm rot}t)\hat{\mathbf{p}} \tag{30}$$

where

$$R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{31}$$

Since the trap is symmetric,

$$\hat{R}_z^\dagger\hat{H}_{\rm cm}\hat{R}_z = \hat{H}_{\rm cm} \tag{32}$$

The rotation term can be represented using a synthetic vector potential,

$$\hat{\mathbf{A}}_\Omega = m\Omega_{\rm rot}\left(-\hat{y}\,\hat{\mathbf{x}} + \hat{x}\,\hat{\mathbf{y}}\right) \tag{33}$$

Then the rotating-frame Hamiltonian becomes

$$\hat{H}_\Omega = \frac{\left(\hat{\mathbf{p}} - \hat{\mathbf{A}}_\Omega\right)^2}{2m} + \frac{1}{2}m\omega_{\rm trap}^2\hat{r}^2 + \hat{W}_{\rm rot} - \hbar\Delta_{\rm eff}\hat{a}^\dagger\hat{a} + \hbar\eta_{\rm eff}f_c\left(\hat{a} + \hat{a}^\dagger\right) \tag{34}$$

where

$$\hat{W}_{\rm rot} = -\frac{1}{2}m\Omega_{\rm rot}^2\hat{r}^2 \tag{35}$$

In this work, an additional harmonic trapping potential is introduced to exactly compensate the centrifugal anti-trapping term:

$$\hat{V}_{\rm comp} = +\frac{1}{2}m\Omega_{\rm rot}^2\hat{r}^2 \tag{36}$$

Therefore,

$$\hat{W}_{\rm rot} + \hat{V}_{\rm comp} = 0 \tag{37}$$

The final compensated rotating-frame Hamiltonian is

$$\hat{H}_\Omega^{\rm (comp)} = \frac{\left(\hat{\mathbf{p}} - \hat{\mathbf{A}}_\Omega\right)^2}{2m} + \frac{1}{2}m\omega_{\rm trap}^2\hat{r}^2 - \hbar\Delta_{\rm eff}\hat{a}^\dagger\hat{a} + \hbar\eta_{\rm eff}f_c\left(\hat{a} + \hat{a}^\dagger\right) \tag{38}$$

We assume that the cavity mode is fixed in the rotating frame, so that the spatial mode function remains time independent after the transformation. We choose

$$f_c(\hat{x},\hat{y}) = \cos(k\hat{x}) \tag{39}$$

and remove the subscript from the vector potential for simplicity:

$$\hat{\mathbf{A}} = m\Omega_{\rm rot}\left(-\hat{y}\,\hat{\mathbf{x}} + \hat{x}\,\hat{\mathbf{y}}\right) \tag{40}$$

Therefore, the final Hamiltonian reads

$$\hat{H} = \frac{\left(\hat{\mathbf{p}} - \hat{\mathbf{A}}\right)^2}{2m} + \frac{1}{2}m\omega_{\rm trap}^2\hat{r}^2 - \hbar\left[\Delta_c - U_0\cos^2(k\hat{x})\right]\hat{a}^\dagger\hat{a} + \hbar\eta_{\rm eff}\cos(k\hat{x})\left(\hat{a} + \hat{a}^\dagger\right) \tag{41}$$

and the Lindblad dissipators are,

$$\mathcal{D}_1\hat{\rho} = \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{42a}$$

$$\mathcal{D}_2\hat{\rho} = \Gamma_0\Big(2\hat{a}\cos(k\hat{x})\hat{\rho}\cos(k\hat{x})\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\cos^2(k\hat{x})\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\cos^2(k\hat{x})\Big) \tag{42b}$$

A. Quantum Harmonic Oscillator Approach

One can rewrite the final Hamiltonian in Eq. (41) by distributing the vector potential, $\hat{\mathbf{A}}$. After algebraic manipulations the Hamiltonian reads,

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\left(\Omega_{\rm rot}^2 + \omega_{\rm trap}^2\right)\hat{r}^2 \\ &- \Omega_{\rm rot}\left(\hat{p}_y\hat{x} - \hat{p}_x\hat{y}\right) \\ &- \hbar\left(\Delta_c - U_0\cos^2(k\hat{x})\right)\hat{a}^\dagger\hat{a} \\ &+ \hbar\eta_{\rm eff}\left(\hat{a}^\dagger + \hat{a}\right)\cos(k\hat{x}) \end{aligned} \tag{43}$$

After substituting $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2$, $\hat{r}^2 = \hat{x}^2 + \hat{y}^2$ and $\omega_o = \sqrt{\Omega_{\rm rot}^2 + \omega_{\rm trap}^2}$ Hamiltonian converts into,

$$\begin{aligned}\hat{H} &= \left[\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega_o^2\hat{x}^2\right] + \left[\frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega_o^2\hat{y}^2\right] \\ &- \Omega_{\rm rot}\left(\hat{p}_y\hat{x} - \hat{p}_x\hat{y}\right) \\ &- \hbar\left(\Delta_c - U_0\cos^2(k\hat{x})\right)\hat{a}^\dagger\hat{a} \\ &+ \hbar\eta_{\rm eff}\left(\hat{a}^\dagger + \hat{a}\right)\cos(k\hat{x}) \end{aligned} \tag{44}$$

As it can be seen from Eq. (44), one can define harmonic oscillator creation and annihilation operators for x and y directions. The mentioned operators are defined as [5]:

$$\begin{aligned}\hat{b}_x &= \sqrt{\frac{m\omega_o}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega_o}\hat{p}_x\right) \\ \hat{b}_y &= \sqrt{\frac{m\omega_o}{2\hbar}}\left(\hat{y} + \frac{i}{m\omega_o}\hat{p}_y\right)\end{aligned} \tag{45}$$

Therefore x operator reads,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right) \tag{46}$$

Via substituting Eqs. (45) and (46) to Eq. (44) Hamiltonian can be rewritten as,

$$\begin{aligned}\hat{H} &= \hbar\omega_o\left(\hat{b}_x^\dagger\hat{b}_x + \frac{1}{2}\right) + \hbar\omega_o\left(\hat{b}_y^\dagger\hat{b}_y + \frac{1}{2}\right) \\ &- \hbar\left(\Delta_c - U_0\cos^2\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right]\right)\hat{a}^\dagger\hat{a} \\ &- i\hbar\Omega_{\rm rot}\left(\hat{b}_x\hat{b}_y^\dagger - \hat{b}_x^\dagger\hat{b}_y\right) \\ &+ \hbar\eta_{\rm eff}\left(\hat{a}^\dagger + \hat{a}\right)\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right]\end{aligned} \tag{47}$$

where $\omega_r = \hbar k^2/2m$. One can also rewrite the Lindblad dissipators by substituting ladder operator representation of $\hat{x}$ into Eq. (18),

$$\mathcal{D}_1\hat{\rho} = \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{48a}$$

$$\begin{aligned}\mathcal{D}_2\hat{\rho} = \Gamma_0\Bigg(&2\hat{a}\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right]\hat{\rho} \\ &\times\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right]\hat{a}^\dagger \\ &- \hat{a}^\dagger\hat{a}\cos^2\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right]\hat{\rho} \\ &- \hat{\rho}\hat{a}^\dagger\hat{a}\cos^2\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right]\Bigg)\end{aligned} \tag{48b}$$

This approach allows us to consider the system as three highly coupled quantum harmonic oscillators.

B. Diagonalization of Mechanical Part of the Hamiltonian

Mechanical part of the Hamiltonian in Eq. (47) can be written as,

$$\begin{aligned}\hat{H}_{\rm mech} &= \hbar\omega_o\left(\hat{b}_x^\dagger\hat{b}_x + \frac{1}{2}\right) + \hbar\omega_o\left(\hat{b}_y^\dagger\hat{b}_y + \frac{1}{2}\right) \\ &- i\hbar\Omega_{\rm rot}\left(\hat{b}_x\hat{b}_y^\dagger - \hat{b}_x^\dagger\hat{b}_y\right) \\ &= \hbar\begin{pmatrix}\hat{b}_x^\dagger & \hat{b}_y^\dagger\end{pmatrix}\begin{pmatrix}\omega_o & i\Omega_{\rm rot} \\ -i\Omega_{\rm rot} & \omega_o\end{pmatrix}\begin{pmatrix}\hat{b}_x \\ \hat{b}_y\end{pmatrix} + \hbar\omega_o\end{aligned} \tag{49}$$

Diagonalization of this Hamiltonian yields two harmonic oscillators with ladder operators and corresponding frequencies,

$$\hat{b}_\pm = \frac{1}{\sqrt{2}}\left(\hat{b}_x \mp i\hat{b}_y\right) \tag{50}$$

$$\omega_\pm = \omega_o \pm \Omega_{\rm rot} \tag{51}$$

For compact notation in the diagonalized basis, we define

$$\hat{X}_d = \sqrt{\frac{\omega_r}{2\omega_o}}\left(\hat{b}_+^\dagger + \hat{b}_+ + \hat{b}_-^\dagger + \hat{b}_-\right) \tag{52}$$

$$\hat{C}_d = \cos\left(\hat{X}_d\right) \tag{53}$$

Then the full Hamiltonian becomes

$$\begin{aligned}\hat{H}_d &= \hbar\omega_+\left(\hat{b}_+^\dagger\hat{b}_+ + \frac{1}{2}\right) + \hbar\omega_-\left(\hat{b}_-^\dagger\hat{b}_- + \frac{1}{2}\right) \\ &- \hbar\left(\Delta_c - U_0\hat{C}_d^2\right)\hat{a}^\dagger\hat{a} + \hbar\eta_{\rm eff}\left(\hat{a}^\dagger + \hat{a}\right)\hat{C}_d\end{aligned} \tag{54}$$

Now we get rid of the coupling term between mechanical oscillators by diagonalizing the mechanical part. The Lindblad dissipators then reads as,

$$\mathcal{D}_1\hat{\rho} = \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{55a}$$

$$\mathcal{D}_2\hat{\rho} = \Gamma_0\left(2\hat{a}\hat{C}_d\hat{\rho}\hat{C}_d\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{C}_d^2\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{C}_d^2\right) \tag{55b}$$

This diagonalized Hamiltonian and corresponding Lindblad dissipators will be helpful simplifying the parity symmetry discussion.

C. Stationary State Analysis for Quantum Harmonic Oscillator Approach

For the quantum harmonic oscillator approach, the Hamiltonian and dissipator are

$$\begin{aligned}\hat{H} &= \hbar\omega_o\left(\hat{b}_x^\dagger\hat{b}_x+\frac{1}{2}\right)+\hbar\omega_o\left(\hat{b}_y^\dagger\hat{b}_y+\frac{1}{2}\right)\\ &\quad-\hbar\left(\Delta_c-U_0\cos^2\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right]\right)\hat{a}^\dagger\hat{a}\\ &\quad-i\hbar\Omega_{\rm rot}\left(\hat{b}_x\hat{b}_y^\dagger-\hat{b}_x^\dagger\hat{b}_y\right)\\ &\quad+\hbar\eta_{\rm eff}\left(\hat{a}^\dagger+\hat{a}\right)\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right]\end{aligned} \tag{56}$$

and

$$\mathcal{D}_1\hat{\rho}=\kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{57a}$$

$$\begin{aligned}\mathcal{D}_2\hat{\rho}=\Gamma_0\Bigg(&2\hat{a}\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right]\hat{\rho}\\ &\times\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right]\hat{a}^\dagger\\ &-\hat{a}^\dagger\hat{a}\cos^2\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right]\hat{\rho}\\ &-\hat{\rho}\hat{a}^\dagger\hat{a}\cos^2\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right]\Bigg)\end{aligned} \tag{57b}$$

where

$$\omega_r=\frac{\hbar k^2}{2m}\qquad\omega_o=\sqrt{\Omega_{\rm rot}^2+\omega_{\rm trap}^2} \tag{58}$$

The master equation is

$$\frac{d\hat{\rho}}{dt}=\frac{i}{\hbar}\left[\hat{\rho},\hat{H}\right]+\sum_j\mathcal{D}_j\hat{\rho}=\mathcal{L}\left(\hat{\rho}\right) \tag{59}$$

To find the stationary state of the quantum system, we solve

$$\mathcal{L}\left(\hat{\rho}\right)=0 \tag{60}$$

For compact notation, we define

$$\hat{C}=\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger+\hat{b}_x\right)\right] \tag{61}$$

$$\hat{n}_c=\hat{a}^\dagger\hat{a}\qquad\hat{n}_x=\hat{b}_x^\dagger\hat{b}_x\qquad\hat{n}_y=\hat{b}_y^\dagger\hat{b}_y \tag{62}$$

$$\hat{K}=\hat{b}_x\hat{b}_y^\dagger-\hat{b}_x^\dagger\hat{b}_y \tag{63}$$

Then the stationary Liouvillian equation can be written explicitly as

$$\begin{aligned}\mathcal{L}\left(\hat{\rho}\right)&=i\omega_o\left[\left(\hat{\rho}\hat{n}_x-\hat{n}_x\hat{\rho}\right)+\left(\hat{\rho}\hat{n}_y-\hat{n}_y\hat{\rho}\right)\right]\\ &\quad-i\Delta_c\left(\hat{\rho}\hat{n}_c-\hat{n}_c\hat{\rho}\right)\\ &\quad+iU_0\left(\hat{\rho}\hat{C}^2\hat{n}_c-\hat{C}^2\hat{n}_c\hat{\rho}\right)\\ &\quad+\Omega_{\rm rot}\left[\hat{\rho}\hat{K}-\hat{K}\hat{\rho}\right]\\ &\quad+i\eta_{\rm eff}\left[\hat{\rho}\left(\hat{a}^\dagger+\hat{a}\right)\hat{C}-\left(\hat{a}^\dagger+\hat{a}\right)\hat{C}\hat{\rho}\right]\\ &\quad+\kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &\quad+\Gamma_0\left(2\hat{a}\hat{C}\hat{\rho}\hat{C}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{C}^2\hat{\rho}-\hat{\rho}\hat{C}^2\hat{a}^\dagger\hat{a}\right)=0\end{aligned} \tag{64}$$

For the diagonalized mechanical Hamiltonian, we use the compact operators defined in Eqs. (52) and (53). The number operators in the circular basis are

$$\hat{n}_+=\hat{b}_+^\dagger\hat{b}_+\qquad\hat{n}_-=\hat{b}_-^\dagger\hat{b}_- \tag{65}$$

Then the stationary Liouvillian equation in the diagonalized basis becomes

$$\begin{aligned}\mathcal{L}_d\left(\hat{\rho}\right)&=i\omega_+\left(\hat{\rho}\hat{n}_+-\hat{n}_+\hat{\rho}\right)+i\omega_-\left(\hat{\rho}\hat{n}_--\hat{n}_-\hat{\rho}\right)\\ &\quad-i\Delta_c\left(\hat{\rho}\hat{n}_c-\hat{n}_c\hat{\rho}\right)+iU_0\left(\hat{\rho}\hat{C}_d^2\hat{n}_c-\hat{C}_d^2\hat{n}_c\hat{\rho}\right)\\ &\quad+i\eta_{\rm eff}\left[\hat{\rho}\left(\hat{a}^\dagger+\hat{a}\right)\hat{C}_d-\left(\hat{a}^\dagger+\hat{a}\right)\hat{C}_d\hat{\rho}\right]\\ &\quad+\kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &\quad+\Gamma_0\left(2\hat{a}\hat{C}_d\hat{\rho}\hat{C}_d\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{C}_d^2\hat{\rho}-\hat{\rho}\hat{C}_d^2\hat{a}^\dagger\hat{a}\right)=0\end{aligned} \tag{66}$$

Equation (66) is equivalent to Eq. (64); it is only written in the circular mechanical basis where the mechanical Hamiltonian is diagonal.

This linear equation can be solved for $\hat{\rho}$ to find the steady states of the system.

The simulation results shows that 2 distinct steady states arises depending on the initial state phonon number and $\Omega_{\rm rot}$ such that,

$$\begin{aligned}&\Omega_{\rm rot}=0\Rightarrow n_x=2k,2k+1\\ &\Omega_{\rm rot}\neq0\Rightarrow n_x+n_y=2k,2k+1\end{aligned} \tag{67}$$

D. Parity Symmetry of Liouvillian

The appearance of two distinct steady states can be understood from the parity symmetry of the Liouvillian. We first define the parity operators

$$\hat{\Pi}_x=e^{i\pi\hat{b}_x^\dagger\hat{b}_x} \tag{68}$$

$$\hat{\Pi}_{xy}=e^{i\pi\left(\hat{b}_x^\dagger\hat{b}_x+\hat{b}_y^\dagger\hat{b}_y\right)} \tag{69}$$

Their action on the Fock states is

$$\hat{\Pi}_x\left|n_x\right\rangle=(-1)^{n_x}\left|n_x\right\rangle \tag{70}$$

$$\hat{\Pi}_{xy}\left|n_x,n_y\right\rangle=(-1)^{n_x+n_y}\left|n_x,n_y\right\rangle \tag{71}$$

Therefore, the Hilbert space can be separated into even and odd parity sectors.

The action of the parity operators on the annihilation operators is

$$\hat{\Pi}_x^\dagger \hat{b}_x \hat{\Pi}_x = -\hat{b}_x \qquad \hat{\Pi}_x^\dagger \hat{b}_y \hat{\Pi}_x = \hat{b}_y \tag{72}$$

$$\hat{\Pi}_{xy}^\dagger \hat{b}_x \hat{\Pi}_{xy} = -\hat{b}_x \qquad \hat{\Pi}_{xy}^\dagger \hat{b}_y \hat{\Pi}_{xy} = -\hat{b}_y \tag{73}$$

Since the cavity coupling depends on

$$\cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right] \tag{74}$$

it is invariant under the transformation $\hat{b}_x \to -\hat{b}_x$. Thus, the optical potential and the pump term do not mix even and odd x-parity sectors.

For $\Omega_{\rm rot} = 0$, the Hamiltonian does not contain the coupling term between x and y directions. Therefore, the relevant parity symmetry is the x-parity symmetry:

$$\left[\hat{H}, \hat{\Pi}_x\right] = 0 \qquad \Omega_{\rm rot} = 0 \tag{75}$$

The corresponding jump operators are also invariant under $\hat{\Pi}_x$, since

$$\left[\hat{a}, \hat{\Pi}_x\right] = 0 \qquad \left[\hat{a}\hat{C}, \hat{\Pi}_x\right] = 0 \tag{76}$$

where

$$\hat{C} = \cos\left[\sqrt{\frac{\omega_r}{\omega_o}}\left(\hat{b}_x^\dagger + \hat{b}_x\right)\right] \tag{77}$$

Hence, $\hat{\Pi}_x$ is a strong symmetry of the Liouvillian for $\Omega_{\rm rot} = 0$.

For $\Omega_{\rm rot} \neq 0$, the Hamiltonian contains the rotational coupling

$$\hat{K} = \hat{b}_x \hat{b}_y^\dagger - \hat{b}_x^\dagger \hat{b}_y \tag{78}$$

Under $\hat{\Pi}_x$, this term changes sign, so the x-parity symmetry is broken:

$$\hat{\Pi}_x^\dagger \hat{K} \hat{\Pi}_x = -\hat{K} \tag{79}$$

However, under the combined parity operator $\hat{\Pi}_{xy}$, both $\hat{b}_x$ and $\hat{b}_y$ change sign, and the rotational coupling remains invariant:

$$\hat{\Pi}_{xy}^\dagger \hat{K} \hat{\Pi}_{xy} = \hat{K} \tag{80}$$

Therefore, for nonzero rotation the relevant parity symmetry is the combined parity symmetry:

$$\left[\hat{H}, \hat{\Pi}_{xy}\right] = 0 \qquad \Omega_{\rm rot} \neq 0 \tag{81}$$

The jump operators also satisfy

$$\left[\hat{a}, \hat{\Pi}_{xy}\right] = 0 \qquad \left[\hat{a}\hat{C}, \hat{\Pi}_{xy}\right] = 0 \tag{82}$$

Thus, $\hat{\Pi}_{xy}$ is a strong symmetry of the Liouvillian for $\Omega_{\rm rot} \neq 0$.

The same symmetry structure can also be understood in the diagonalized basis. In this basis, the joint parity operator is

$$\hat{\Pi}_d = e^{i\pi\left(\hat{b}_+^\dagger \hat{b}_+ + \hat{b}_-^\dagger \hat{b}_-\right)} \tag{83}$$

Since the circular transformation is unitary,

$$\hat{b}_+^\dagger \hat{b}_+ + \hat{b}_-^\dagger \hat{b}_- = \hat{b}_x^\dagger \hat{b}_x + \hat{b}_y^\dagger \hat{b}_y \tag{84}$$

and therefore

$$\hat{\Pi}_d = \hat{\Pi}_{xy} \tag{85}$$

This means that the diagonalized Hamiltonian always has the joint parity symmetry

$$\left[\hat{H}_d, \hat{\Pi}_d\right] = 0 \tag{86}$$

for all values of $\Omega_{\rm rot}$.

The action of $\hat{\Pi}_d$ on the diagonalized ladder operators is

$$\hat{\Pi}_d^\dagger \hat{b}_\pm \hat{\Pi}_d = -\hat{b}_\pm \tag{87}$$

Therefore, the diagonalized cosine operator transforms as

$$\begin{aligned}\hat{\Pi}_d^\dagger \hat{C}_d \hat{\Pi}_d &= \cos\left[-\sqrt{\frac{\omega_r}{2\omega_o}}\left(\hat{b}_+^\dagger + \hat{b}_+ + \hat{b}_-^\dagger + \hat{b}_-\right)\right] \\ &= \hat{C}_d \end{aligned} \tag{88}$$

Thus, the Hamiltonian and jump operators in the diagonalized basis are invariant under $\hat{\Pi}_d$.

However, this does not mean that the separate x-parity symmetry is lost when $\Omega_{\rm rot} = 0$. The transformation to the circular basis is still valid at $\Omega_{\rm rot} = 0$, but the x-parity symmetry is represented differently. The x-parity operator is

$$\hat{\Pi}_x = e^{i\pi \hat{b}_x^\dagger \hat{b}_x} \tag{89}$$

and it acts on the original ladder operators as

$$\hat{\Pi}_x^\dagger \hat{b}_x \hat{\Pi}_x = -\hat{b}_x \qquad \hat{\Pi}_x^\dagger \hat{b}_y \hat{\Pi}_x = \hat{b}_y \tag{90}$$

Using the circular definitions in Eq. (50), this gives

$$\hat{\Pi}_x^\dagger \hat{b}_+ \hat{\Pi}_x = -\hat{b}_- \tag{91}$$

$$\hat{\Pi}_x^\dagger \hat{b}_- \hat{\Pi}_x = -\hat{b}_+ \tag{92}$$

Therefore, in the diagonalized basis, the x-parity operation is not a simple sign flip of one mode. Instead, it exchanges the two circular modes and adds a minus sign.

The interaction term remains invariant under this operation because the argument of the cosine changes sign:

$$\hat{b}_+^\dagger + \hat{b}_+ + \hat{b}_-^\dagger + \hat{b}_- \xrightarrow{\hat{\Pi}_x} -\left(\hat{b}_+^\dagger + \hat{b}_+ + \hat{b}_-^\dagger + \hat{b}_-\right) \tag{93}$$

Since the cosine is an even function, the optical part of the Hamiltonian and the jump operators remain invariant.

The mechanical part in the diagonalized basis is

$$\hat{H}_{\mathrm{mech},d} = \hbar\left(\omega_o + \Omega_{\mathrm{rot}}\right)\hat{b}_+^\dagger\hat{b}_+ + \hbar\left(\omega_o - \Omega_{\mathrm{rot}}\right)\hat{b}_-^\dagger\hat{b}_- \quad (94)$$

where the constant zero-point energy terms are omitted. Under $\hat{\Pi}_x$, the number operators are exchanged:

$$\hat{b}_+^\dagger\hat{b}_+ \leftrightarrow \hat{b}_-^\dagger\hat{b}_- \quad (95)$$

Thus,

$$\begin{aligned}\hat{\Pi}_x^\dagger\hat{H}_{\mathrm{mech},d}\hat{\Pi}_x = {} & \hbar\left(\omega_o + \Omega_{\mathrm{rot}}\right)\hat{b}_-^\dagger\hat{b}_- \\ & + \hbar\left(\omega_o - \Omega_{\mathrm{rot}}\right)\hat{b}_+^\dagger\hat{b}_+\end{aligned} \quad (96)$$

This is equal to the original $\hat{H}_{\mathrm{mech},d}$ only when $\Omega_{\mathrm{rot}} = 0$ because only then the two circular modes are degenerate. Therefore, the circular basis does not remove the x-parity symmetry at zero rotation. It only disguises it as an exchange symmetry between the $+$ and $-$ modes.

This also resolves the apparent contradiction that the diagonalized Hamiltonian always satisfies $\hat{\Pi}_d = \hat{\Pi}_{xy}$, while the original Hamiltonian satisfies $\hat{\Pi}_x$ only for $\Omega_{\mathrm{rot}} = 0$. The joint parity symmetry is present for all values of Ω_{rot}. The additional x-parity symmetry appears only at $\Omega_{\mathrm{rot}} = 0$, where the two circular modes become degenerate and the exchange symmetry between them becomes exact. For nonzero rotation, the degeneracy is lifted because $\omega_+ \neq \omega_-$, so the exchange symmetry is broken and only the joint parity symmetry remains.

More formally, if $\hat{\Pi}$ denotes the relevant parity operator, the corresponding parity superoperator is

$$\mathcal{P}\hat{\rho} = \hat{\Pi}\hat{\rho}\hat{\Pi}^\dagger \quad (97)$$

The Liouvillian symmetry condition is

$$\mathcal{L}\left(\hat{\Pi}\hat{\rho}\hat{\Pi}^\dagger\right) = \hat{\Pi}\mathcal{L}\left(\hat{\rho}\right)\hat{\Pi}^\dagger \quad (98)$$

or equivalently

$$[\mathcal{L}, \mathcal{P}] = 0 \quad (99)$$

Because of this symmetry, the Liouvillian does not mix density matrices belonging to different parity sectors[1]. The Hilbert space can be written as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \quad (100)$$

where

$$\mathcal{H}_+ = \left\{|\psi\rangle : \hat{\Pi}\,|\psi\rangle = +\,|\psi\rangle\right\} \quad (101)$$

$$\mathcal{H}_- = \left\{|\psi\rangle : \hat{\Pi}\,|\psi\rangle = -\,|\psi\rangle\right\} \quad (102)$$

For $\Omega_{\mathrm{rot}} = 0$, these sectors correspond to even and odd values of n_x. For $\Omega_{\mathrm{rot}} \neq 0$, they correspond to even and odd values of $n_x + n_y$.

As a result, an initial state in one parity sector remains in that sector throughout the time evolution:

$$\hat{\rho}(0) \in \mathcal{H}_\pm \quad \Longrightarrow \quad \hat{\rho}(t) \in \mathcal{H}_\pm \quad (103)$$

Therefore, the steady state depends on the parity sector of the initial state:

$$\hat{\rho}(t) \longrightarrow \hat{\rho}_{\mathrm{ss}}^{(+)} \qquad \text{or} \qquad \hat{\rho}(t) \longrightarrow \hat{\rho}_{\mathrm{ss}}^{(-)} \quad (104)$$

This explains how the same linear master equation can lead to two distinct steady states. The Liouvillian is linear, but because it is block diagonal in the parity sectors, each symmetry sector can have its own stationary solution.

Furthermore, let us consider an arbitrary initial density matrix $\hat{\rho}_0$. The relevant parity operator will be denoted by $\hat{\Pi}$. The corresponding even and odd parity projectors are

$$\hat{P}_\pm = \frac{1}{2}\left(\hat{I} \pm \hat{\Pi}\right) \quad (105)$$

such that the initial parity weights are

$$p_\pm = \mathrm{Tr}\left[\hat{P}_\pm\hat{\rho}_0\right] \quad (106)$$

with

$$p_+ + p_- = 1 \quad (107)$$

Equivalently, using the initial parity expectation value,

$$\langle\hat{\Pi}\rangle_0 = \mathrm{Tr}\left[\hat{\Pi}\hat{\rho}_0\right] \quad (108)$$

the weights can be written as

$$p_\pm = \frac{1 \pm \langle\hat{\Pi}\rangle_0}{2} \quad (109)$$

The initial density matrix can be decomposed into parity blocks as

$$\begin{aligned}\hat{\rho}_0 = {} & \hat{P}_+\hat{\rho}_0\hat{P}_+ + \hat{P}_-\hat{\rho}_0\hat{P}_- \\ & + \hat{P}_+\hat{\rho}_0\hat{P}_- + \hat{P}_-\hat{\rho}_0\hat{P}_+\end{aligned} \quad (110)$$

Because the parity is a strong symmetry of the Liouvillian, the diagonal parity weights are conserved during the time evolution:

$$\frac{d}{dt}\mathrm{Tr}\left[\hat{P}_\pm\hat{\rho}(t)\right] = 0 \quad (111)$$

If each parity sector has one normalized stationary state, denoted by $\hat{\rho}_{ss}^{(+)}$ and $\hat{\rho}_{ss}^{(-)}$, then

$$\mathcal{L}\left(\hat{\rho}_{ss}^{(\pm)}\right) = 0 \quad (112)$$

with

$$\mathrm{Tr}\left[\hat{P}_\pm\hat{\rho}_{ss}^{(\pm)}\right] = 1 \qquad \mathrm{Tr}\left[\hat{P}_\mp\hat{\rho}_{ss}^{(\pm)}\right] = 0 \quad (113)$$

Therefore, after the off diagonal parity coherences decay, the long-time state becomes

$$\hat{\rho}(t \to \infty) = p_+ \hat{\rho}_{ss}^{(+)} + p_- \hat{\rho}_{ss}^{(-)} \tag{114}$$

or equivalently,

$$\hat{\rho}(t \to \infty) = \frac{1 + \langle \hat{\Pi} \rangle_0}{2} \hat{\rho}_{ss}^{(+)} + \frac{1 - \langle \hat{\Pi} \rangle_0}{2} \hat{\rho}_{ss}^{(-)} \tag{115}$$

Since the Liouvillian is linear, this final state is also a stationary state:

$$\begin{aligned} \mathcal{L}\left[p_+ \hat{\rho}_{ss}^{(+)} + p_- \hat{\rho}_{ss}^{(-)}\right] &= p_+ \mathcal{L}\left[\hat{\rho}_{ss}^{(+)}\right] + p_- \mathcal{L}\left[\hat{\rho}_{ss}^{(-)}\right] \\ &= 0 \end{aligned} \tag{116}$$

Thus, the linearity of the master equation does not imply a unique steady state. Instead, because the Liouvillian is block diagonal in the parity sectors, each sector can have its own stationary state, and a general initial state can lead to the parity weighted mixture of these steady states.

For example if the initial state is a superposition of an odd parity state and an even parity state,

$$|\psi_0\rangle = \alpha |\psi_-\rangle + \beta |\psi_+\rangle \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{117}$$

then the corresponding density matrix is

$$\begin{aligned} \hat{\rho}_0 = |\alpha|^2 |\psi_-\rangle \langle \psi_-| &+ |\beta|^2 |\psi_+\rangle \langle \psi_+| \\ &+ \alpha\beta^* |\psi_-\rangle \langle \psi_+| \\ &+ \alpha^*\beta |\psi_+\rangle \langle \psi_-| \end{aligned} \tag{118}$$

Therefore the initial parity weights are

$$p_- = |\alpha|^2 \qquad p_+ = |\beta|^2 \tag{119}$$

and the initial parity expectation value is

$$\langle \hat{\Pi} \rangle_0 = |\beta|^2 - |\alpha|^2 \tag{120}$$

Thus the steady state this initial state will lead is

$$\hat{\rho}_{ss} = |\alpha|^2 \hat{\rho}_{ss}^{(-)} + |\beta|^2 \hat{\rho}_{ss}^{(+)} \tag{121}$$

This approach will be helpful in numerical part of the project. We can construct only the Hilbert subspaces for even and odd parity and for a set of parameters find the two distinct steady states of the system. Then if we want to find the steady state for an arbitrary initial state we can directly use the superposition of previously calculated steady states with appropriate coefficients.

E. Semi-Classical Approach

One can use Ehrenfest Theorem to consider operators as numbers in a semiclassical point of view. The Ehrenfest formula reads as,

$$\frac{d}{dt}\langle \hat{O} \rangle = \frac{i}{\hbar}\left\langle \left[\hat{H}, \hat{O}\right] \right\rangle + \sum_j \left\langle \mathcal{L}_j^\dagger [\hat{O}] \right\rangle + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle \tag{122}$$

Where $\hat{O}$ denotes any arbitrary operator and $\mathrm{Tr}\left[(\mathcal{L}\hat{\rho})\hat{O}\right] = \mathrm{Tr}\left[\hat{\rho}(\mathcal{L}^\dagger \hat{O})\right]$ [6]. We can say that Hamiltonian used has no explicit time dependence so operators do not have any explicit time dependence so $\partial_t \hat{O} = 0$ in general.

One can define $a = \langle \hat{a} \rangle$, $b_x = \langle \hat{b}_x \rangle$ and $b_y = \langle \hat{b}_y \rangle$ for the sake of semiclassical approximation. For compact notation, we define $\lambda = \sqrt{\omega_r/\omega_o}$ so that

$$kx = 2\lambda \,\mathrm{Re}[b_x] \tag{123}$$

By using Eq. (47) in Eq. (122) one can derive the stochastic differential equations describing the system as follows,

$$\begin{aligned} \frac{da}{dt} = \Bigg[&- \left(\kappa + \Gamma_0 \cos^2\left[2\lambda \,\mathrm{Re}[b_x]\right]\right) \\ &+ i\left(\Delta_c - U_0 \cos^2\left[2\lambda \,\mathrm{Re}[b_x]\right]\right) \Bigg] a \\ &- i\eta_{\text{eff}} \cos\left[2\lambda \,\mathrm{Re}[b_x]\right] + \xi_a \end{aligned} \tag{124a}$$

$$\begin{aligned} \frac{db_x}{dt} = &-i\omega_o b_x + \Omega_{\text{rot}} b_y \\ &+ iU_0 \lambda |a|^2 \sin\left[4\lambda \,\mathrm{Re}[b_x]\right] \\ &+ 2i\eta_{\text{eff}} \lambda \,\mathrm{Re}[a] \sin\left[2\lambda \,\mathrm{Re}[b_x]\right] + \xi_{b_x} \end{aligned} \tag{124b}$$

$$\frac{db_y}{dt} = -i\omega_o b_y - \Omega_{\text{rot}} b_x + \xi_{b_y} \tag{124c}$$

Where ξ_i are Langevin noise terms originating from vacuum field input and spontaneous emission of the atom. We can first write the stochastic noise for the cavity field and momentum variables. The general noise correlations are [7]

$$\langle \xi_m^* \xi_n \rangle = \kappa \delta_{nm} + \sum_{\alpha=1}^{N} \Gamma_0 f_m(\mathbf{r}_\alpha) f_n^*(\mathbf{r}_\alpha) \tag{125a}$$

$$\left\langle \xi_{p_i}^{(\alpha)} \xi_n \right\rangle = i\hbar \Gamma_0 \partial_i \mathcal{E}(\mathbf{r}_\alpha) f_n^*(\mathbf{r}_\alpha) \tag{125b}$$

$$\begin{aligned} \left\langle \xi_{p_i}^{(\alpha)} \xi_{p_j}^{(\alpha)} \right\rangle = &\, 2\hbar^2 k_A^2 \Gamma_0 \left|\mathcal{E}(\mathbf{r}_\alpha)\right|^2 \overline{u_i^2} \delta_{ij} \\ &+ \Gamma_0 \hbar^2 \left[\partial_i \mathcal{E}^*(\mathbf{r}_\alpha) \partial_j \mathcal{E}(\mathbf{r}_\alpha) \right. \\ &\left. + \partial_i \mathcal{E}(\mathbf{r}_\alpha) \partial_j \mathcal{E}^*(\mathbf{r}_\alpha)\right] \end{aligned} \tag{125c}$$

Where n, m corresponds to cavity field a, i, j corresponds to spatial modes x, y and α depicts the specific particle. For the single cavity mode and single particle case, we use

$$\mathcal{E}(\mathbf{r}) = a\cos(kx) \qquad f(\mathbf{r}) = \cos(kx) \tag{126}$$

which corresponds to the cavity field and cavity mode function. Therefore, the noise correlations in the momentum and cavity variables become

$$\langle \xi_a^* \xi_a \rangle = \kappa + \Gamma_0 \cos^2(kx)$$

$$= \kappa + \Gamma_0 \cos^2\left[2\lambda\,\mathrm{Re}[b_x]\right] \tag{127a}$$

$$\begin{aligned}\langle \xi_{p_x}\xi_a\rangle &= -i\hbar k\Gamma_0 a\sin(2kx)\\ &= -i\hbar k\Gamma_0 a\sin\left[4\lambda\,\mathrm{Re}[b_x]\right]\end{aligned} \tag{127b}$$

$$\begin{aligned}\langle \xi_{p_x}\xi_{p_x}\rangle &= 2\hbar^2k^2\Gamma_0|a|^2\left[\overline{u_x^2}\cos^2(kx)+\sin^2(kx)\right]\\ &= 2\hbar^2k^2\Gamma_0|a|^2\left[\overline{u_x^2}\cos^2\left[2\lambda\,\mathrm{Re}[b_x]\right]\right.\\ &\qquad \left.+\sin^2\left[2\lambda\,\mathrm{Re}[b_x]\right]\right]\end{aligned} \tag{127c}$$

$$\begin{aligned}\left\langle \xi_{p_y}\xi_{p_y}\right\rangle &= 2\hbar^2k^2\Gamma_0|a|^2\overline{u_y^2}\cos^2(kx)\\ &= 2\hbar^2k^2\Gamma_0|a|^2\overline{u_y^2}\cos^2\left[2\lambda\,\mathrm{Re}[b_x]\right]\end{aligned} \tag{127d}$$

Now we convert the momentum noise correlations into annihilation-operator noise correlations. From the definition of the semiclassical oscillator variables,

$$b_i = \sqrt{\frac{m\omega_o}{2\hbar}}x_i + \frac{i}{\sqrt{2\hbar m\omega_o}}p_i \tag{128}$$

one can see that the stochastic part of b_i comes only from the stochastic momentum kick. Therefore,

$$\xi_{b_i} = \frac{i}{\sqrt{2\hbar m\omega_o}}\xi_{p_i} \tag{129}$$

This means that the cross-correlation with the cavity noise transforms as

$$\langle \xi_{b_i}\xi_a\rangle = \frac{i}{\sqrt{2\hbar m\omega_o}}\langle \xi_{p_i}\xi_a\rangle \tag{130}$$

and the mechanical noise correlations transform as

$$\left\langle \xi_{b_i}\xi_{b_j}\right\rangle = -\frac{1}{2\hbar m\omega_o}\left\langle \xi_{p_i}\xi_{p_j}\right\rangle \tag{131}$$

Using $\omega_r = \hbar k^2/2m$, the nonzero noise correlations in the variables a, b_x, and b_y are

$$\langle \xi_a^*\xi_a\rangle = \kappa + \Gamma_0\cos^2\left[2\lambda\,\mathrm{Re}[b_x]\right] \tag{132a}$$

$$\langle \xi_{b_x}\xi_a\rangle = \Gamma_0\lambda a\sin\left[4\lambda\,\mathrm{Re}[b_x]\right] \tag{132b}$$

$$\begin{aligned}\langle \xi_{b_x}\xi_{b_x}\rangle &= -2\Gamma_0\frac{\omega_r}{\omega_o}|a|^2\left[\overline{u_x^2}\cos^2\left[2\lambda\,\mathrm{Re}[b_x]\right]\right.\\ &\qquad\left.+\sin^2\left[2\lambda\,\mathrm{Re}[b_x]\right]\right]\end{aligned} \tag{132c}$$

$$\left\langle \xi_{b_y}\xi_{b_y}\right\rangle = -2\Gamma_0\frac{\omega_r}{\omega_o}|a|^2\overline{u_y^2}\cos^2\left[2\lambda\,\mathrm{Re}[b_x]\right] \tag{132d}$$

Where $\bar{u}_i$ are the averaged projection of the spontaneous emission on corresponding axes. The negative sign in Eqs. (132c) and (132d) is not a negative physical diffusion. It appears because the annihilation variables contain the momentum with a factor of i. In an equivalent real phase-space representation, the corresponding momentum diffusion terms are positive.

The Langevin equation can be written in differential form as [8]

$$d\mathbf{u} = \mathbf{F}(\mathbf{u})dt + \mathbf{B}(\mathbf{u})d\mathbf{W} \tag{133}$$

Here

$$\mathbf{u} = (a, b_x, b_y)^T \tag{134}$$

and $d\mathbf{W}$ is a Wiener increment. The number of independent Wiener increments is chosen such that the matrix $\mathbf{B}$ satisfies the following relation to the diffusion matrix $\mathbf{D}$,

$$\mathbf{D} = \mathbf{B}\mathbf{B}^T \tag{135}$$

and the noise correlations satisfy

$$\langle dW_i dW_j\rangle = \delta_{ij}dt \qquad \langle \xi_i(t)\xi_j(t')\rangle = D_{ij}\delta(t-t') \tag{136}$$

These relations provide a convenient way to convert the Langevin equations into Wiener process form, which is more suitable for numerical implementation. The equations above can be implemented directly as complex stochastic differential equations. If a numerical solver requires real-valued variables, the same system can be separated into real and imaginary parts only at the implementation stage.

Other Approximation Techniques

1. Positive-P representation
2. Truncated Wigner Approximation (Possibly giving the same equations)
3. Semiclassical Initial Value Representation

II. SIMULATION RESULTS

The dimensionless variables defined as $\mathcal{E} = E/\hbar\omega_r$ and $\tau = \omega_r t$ where $\omega_r = \hbar k^2/2m$. In the numerical simulations all equations are reformulated and the results are plotted using mentioned dimensionless variables.

A. Stationary State Analysis of Linbladian for Various Set of Parameters and Initial States

B. Time Dynamics of the Chosen Set of Parameters and Initial States

C. Comparison with Semiclassical Approach

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